Example 1: Finding the Solutions of a Trigonometric Equation

Solve the equation: $3\sin x - 2 = 5\sin x - 1$.

Solution The equation contains a single trigonometric function, $\sin x$.

Step 1 **Isolate the function on one side of the equation.** We can solve for $\sin x$ by collecting terms with $\sin x$ on the left side and constant terms on the right side.

$3\sin x - 2 = 5\sin x - 1$	This is the given equation.
$3\sin x - 5\sin x - 2 = 5\sin x - 5\sin x - 1$	Subtract 5 sin x from both sides.
$-2\sin x - 2 = -1$	Simplify.
$-2\sin x = 1$	Add 2 to both sides.
$\sin x = -\frac{1}{2}$	Divide both sides by −2 and solve for sin x.

Step 2 **Solve for the variable.** We must solve for x in $\sin x = -\frac{1}{2}$. Because $\sin\frac{\pi}{6} = \frac{1}{2}$, the solutions of $\sin x = -\frac{1}{2}$ in $[0, 2\pi)$ are

$$x = \pi + \frac{\pi}{6} = \frac{6\pi}{6} + \frac{\pi}{6} = \frac{7\pi}{6} \qquad x = 2\pi - \frac{\pi}{6} = \frac{12\pi}{6} - \frac{\pi}{6} = \frac{11\pi}{6}.$$

The sine is negative in quadrant III.

The sine is negative in quadrant IV.

Because the period of the sine function is 2π, the solutions of the equation are given by

$$x = \frac{7\pi}{6} + 2n\pi \quad \text{and} \quad x = \frac{11\pi}{6} + 2n\pi,$$

where n is any integer.

Example 2: Solving an Equation with a Multiple Angle

Solve the equation: $\sin\frac{x}{2} = \frac{\sqrt{3}}{2}, 0 \leq x < 2\pi.$

Solution The period of the sine function is 2π. In the interval $[0, 2\pi)$, there are two values at which the sine function is $\frac{\sqrt{3}}{2}$. One of these values is $\frac{\pi}{3}$. The sine is positive in quadrant II; thus, the other value is $\pi - \frac{\pi}{3}$, or $\frac{2\pi}{3}$. This means that $\frac{x}{2} = \frac{\pi}{3}$ or $\frac{x}{2} = \frac{2\pi}{3}$. Because the period is 2π, all the solutions of $\sin\frac{x}{2} = \frac{\sqrt{3}}{2}$ are given by

$$\frac{x}{2} = \frac{\pi}{3} + 2n\pi \quad \text{or} \quad \frac{x}{2} = \frac{2\pi}{3} + 2n\pi. \quad \text{n is any integer.}$$

$$x = \frac{2\pi}{3} + 4n\pi \qquad x = \frac{4\pi}{3} + 4n\pi. \quad \text{Multiply both sides by 2 and solve for x.}$$

We see that $x = \frac{2\pi}{3} + 4n\pi$ or $x = \frac{4\pi}{3} + 4n\pi$. If $n = 0$, we obtain $x = \frac{2\pi}{3}$ from the first equation and $x = \frac{4\pi}{3}$ from the second equation. If we let $n = 1$, we are adding $4 \cdot 1 \cdot \pi$, or 4π, to $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$. These values of x exceed 2π. Thus, in the interval $[0, 2\pi)$, the only solutions of $\sin\frac{x}{2} = \frac{\sqrt{3}}{2}$ are $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$.

Example 3: Solving a Trigonometric Equation Quadratic in Form

Solve the equation: $2\cos^2 x + \cos x - 1 = 0, \quad 0 \le x < 2\pi.$

Solution The given equation is in quadratic form $2u^2 + u - 1 = 0$ with $u = \cos x$. Let us attempt to solve the equation by factoring.

$$2\cos^2 x + \cos x - 1 = 0$$ This is the given equation.

$$(2\cos x - 1)(\cos x + 1) = 0$$ Factor. Notice that $2u^2 + u - 1$ factors as $(2u - 1)(u + 1)$.

$$2\cos x - 1 = 0 \quad \text{or} \quad \cos x + 1 = 0$$ Set each factor equal to 0.

$$2\cos x = 1 \qquad \cos x = -1$$ Solve for cos x.

$$\cos x = \frac{1}{2}$$

$$x = \frac{\pi}{3} \qquad x = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3} \qquad x = \pi$$ Solve each equation for x, $0 \le x < 2\pi$.

The cosine is positive in quadrants I and IV.

The solutions in the interval $[0, 2\pi)$ are $\frac{\pi}{3}$, π, and $\frac{5\pi}{3}$.

Example 4: Using Factoring to Separate Different Functions

Solve the equation: $\tan x \sin^2 x = 3\tan x, \quad 0 \le x < 2\pi.$

Solution Move all terms to one side and obtain zero on the other side.

$$\tan x \sin^2 x = 3\tan x$$ This is the given equation.

$$\tan x \sin^2 x - 3\tan x = 0$$ Subtract 3 tan x from both sides.

We now have $\tan x \sin^2 x - 3\tan x = 0$, which contains both tangent and sine functions. Use factoring to separate the two functions.

$$\tan x(\sin^2 x - 3) = 0$$ Factor out tan x from the two terms on the left side.

$$\tan x = 0 \quad \text{or} \quad \sin^2 x - 3 = 0$$ Set each factor equal to 0.

$$x = 0 \quad x = \pi \qquad \sin^2 x = 3$$ Solve for x.

$$\sin x = \pm\sqrt{3}$$

This equation has no solution because $\sin x$ cannot be greater than 1 or less than −1.

The solutions in the interval $[0, 2\pi)$ are 0 and π.

Example 5: Using an Identity to Solve a Trigonometric Equation

Solve the equation: $2\cos^2 x + \cos x - 1 = 0, \quad 0 \le x < 2\pi.$

Solution The given equation is in quadratic form $2u^2 + u - 1 = 0$ with $u = \cos x$. Let us attempt to solve the equation by factoring.

$$2\cos^2 x + \cos x - 1 = 0$$
This is the given equation.

$$(2\cos x - 1)(\cos x + 1) = 0$$
Factor. Notice that $2u^2 + u - 1$ factors as $(2u - 1)(u + 1)$.

$$2\cos x - 1 = 0 \quad \text{or} \quad \cos x + 1 = 0$$
Set each factor equal to 0.

$$2\cos x = 1 \qquad \cos x = -1$$
Solve for cos x.

$$\cos x = \frac{1}{2}$$

$$x = \frac{\pi}{3} \qquad x = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3} \qquad x = \pi$$
Solve each equation for x, $0 \le x < 2\pi$.

The cosine is positive in quadrants I and IV.

The solutions in the interval $[0, 2\pi)$ are $\frac{\pi}{3}$, π, and $\frac{5\pi}{3}$.

Practice:

A. Solve each equation on the interval $[0, 2\pi)$.

1. $2\sin x + \sqrt{3} = 0$
2. $\cos 4x = -\frac{\sqrt{3}}{2}$
3. $2\sin^2 x = \sin x + 3$
4. $(\tan x - 1)(\cos x + 1) = 0$

B. Use a calculator to solve each equation on the interval $[0, 2\pi)$. Correct to four decimal places).

5. $\sin x = 0.8246$
6. $\sin x = 0.7392$