The Law of Cosines: Triangles are a fundamental geometrical shape. Triangles appear within several disciplines, some of which are architecture, engineering, astronomy, and chemistry. This is why mathematicians have studied them and consequently have several relations to enumerate their sides and angles. Here is one such relation. These are the equations collectively called *The Law of Cosines*.

$$c^2 = a^2 + b^2 - 2ab \cdot cosC$$

$$b^2 = a^2 + c^2 - 2ac \cdot cosB$$

$$a^2 = b^2 + c^2 - 2bc \cdot cosA$$

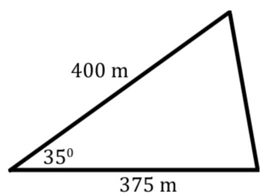
These equations are useful for two cases:

1) When two sides and the angle ***between*** them are known within a triangle.
2) When all three sides of a triangle are known.

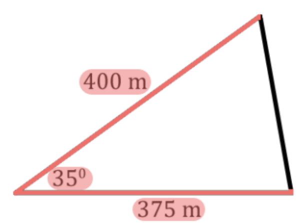
I. Calculating the Length of a Side

- Do you prefer to watch a video walking you through these steps? If not, read through the model example below and copy the problem into your notes.

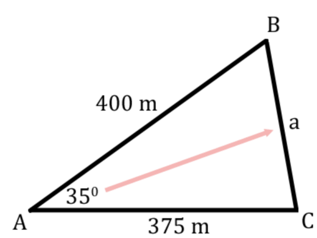
Model Example: Find the length of the missing side in the triangle below using the given information and the law of cosines.



The law of cosines is a helpful tool for this situation because we know a triangle's angle and the sides that are next to it, as emphasized via the colored graphic below.



Now that we know the law of cosines can be used for the problem, we need to label the triangle's angles. The triangle can be arbitrarily labeled with the letters 'A,' 'B,' and 'C.' The side opposite angle-A, our 35-degree angle, is side-a.



Likewise, side-b is 375, side-c is 400. Using this set of information, we have to determine which law of cosines equation is best to use. Since we know angle-A, we should use the formula with angle-A in it. *The side we calculate will always be the side opposite the given angle.*

$$a^2 = b^2 + c^2 - 2bc \cdot cosA$$

Using this equation, we need to plug in the given information within it.

$$a^2 = 375^2 + 400^2 - 2(375)(400) \cdot cos35$$

Using a calculator (or www.desmos.com) this can be used to simplify the right side of the equation.

$$a^2 = 54879.3867$$

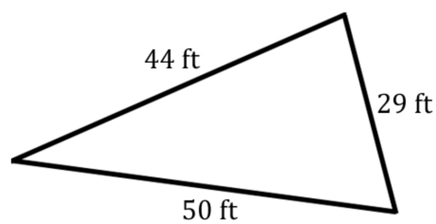
$$\sqrt{a^2} = \sqrt{54879.3867}$$

$$a = 234.3\ m$$

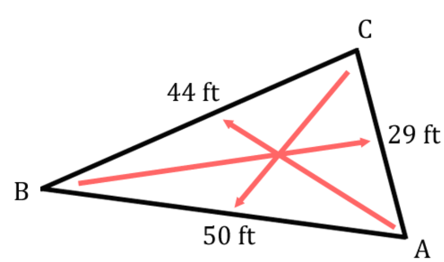
II. **Calculating the Measurement of an Angle**

- Do you prefer to watch a video walking you through the steps? If not, read through the model example below and copy the problem into your notes.

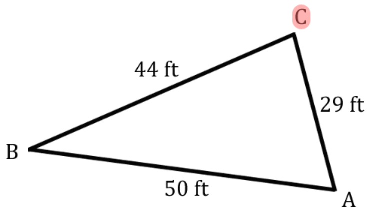
Model Example: Find the measurement of the angle opposite 50 degrees in the triangle below using the given information and the law of cosines.



The given information above is one where we know all three sides of a triangle. Let's label the angles with 'A,' 'B,' and 'C.' Recall, we look to the sides opposite an angle and label them accordingly. For instance, side-c is opposite angle-C. The graphic below demonstrates this labeling of sides.



It is now clear that side-a is 44, side-b is 29, and side-c is 50. Based on our labelling, we are looking for angle C and can use the top equation law of cosines equation.



$$c^2 = a^2 + b^2 - 2ab \cdot cosC$$
$$b^2 = a^2 + c^2 - 2ac \cdot cosB$$
$$a^2 = b^2 + c^2 - 2bc \cdot cosA$$

$$c^2 = a^2 + b^2 - 2ab \cdot cosC$$

Let's plug in the given information: side-a is 44, side-b is 29, and side-c is 50.

$$50^2 = 44^2 + 29^2 - 2(44)(29) \cdot cosC$$

$$2500 = 2777 - 2552 \cdot cosC$$

*Commonly, students make an error at this step. Some students think these two numbers can be subtracted. However, these numbers cannot be combined because they are not like terms. Do not make the mistake of combining these values.

$$2500 = 2777 - 2552 \cdot cosC$$

Instead of combining unlike terms, we have to cancel the 2777 by subtracting that value from both sides of the equation.

$$\begin{aligned} 2500 &= 2777 - 2552 \cdot cosC \\ -2777 & \quad -2777 \end{aligned}$$

$$-277 = -2552 \cdot cosC$$

$$\frac{-277}{-2552} = \frac{-2552 \cdot cosC}{-2552}$$

$$\frac{-277}{-2552} = cosC$$

*Insead of dividing the left side, leave the value as a fraction so that we can avoid rounding a decimal value. It's better to round the value once at the conclusion of our calculations, allowing us to be more precise.

We have nearly solved our equation for angle-C. However, we have the cosine of angle-C. The next step must involve canceling the cosine function. The only way to cancel the cosine function is to use the inverse cosine function, like so.

$$\cos^{-1}\left(\frac{-277}{-2552}\right) = \cos^{-1}(\cos C)$$

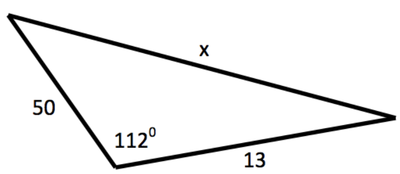
This is what our equation becomes after we cancel the cosine function with its inverse cosine function.

$$\cos^{-1}\left(\frac{-277}{-2552}\right) = C$$

$$83.8 = C$$

PRACTICE: Solve for the missing value in each diagram below using the Law of Cosines.

1.



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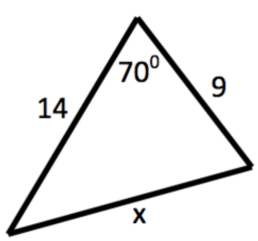

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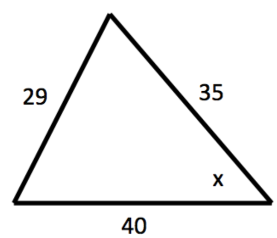

4.